

SOME CHARACTERIZATIONS FOR THE INVOLUTE CURVES IN DUAL SPACE

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ABSTRACT. In this paper, we investigate some characterizations of involute – evolute curves in dual space. Then the relationships between dual frenet frame and darboux vectors of these curves are found.

1. INTRODUCTION

Foundation notion with respect to involute of a curve are given in 3-dimensional Euclidean space IR^3 , [3],[8]. The relationships between frenet frames of involute – evolute curves and some characterizations related to these curves in 3-dimensional Euclidean space IR^3 and Minkowski space IR_1^3 are described by Caliskan and Bilici, [1],[2],[3].

In this study some new characterizations with respect to involute – evolute curves ID^3 in are given.

2. PRELIMINARIES

The set $ID = \left\{ \overset{\Lambda}{a} = a + \varepsilon a^* \mid a, a^* \in IR, \varepsilon^2 = 0 \right\}$ is called dual numbers set by W.K. Clifford (1849-79) as a tool for his geometrical investigations.

Product and addition operations on this set are described respectively,

$$\begin{aligned} (a + \varepsilon a^*) + (b + \varepsilon b^*) &= (a + b) + \varepsilon (a^* + b^*) \\ (a + \varepsilon a^*) \cdot (b + \varepsilon b^*) &= ab + \varepsilon (ab^* + a^*b) \end{aligned}$$

Algebraic construction $(ID, +, \cdot)$ is unit and commutative ring.

Addition and scalar product on $ID^3 = \left\{ \overset{\Lambda}{a} \mid \overset{\Lambda}{a} = \vec{a} + \varepsilon \vec{a}^*, \vec{a}, \vec{a}^* \in IR^3 \right\}$ are described.

$$\begin{aligned} \oplus : ID^3 \times ID^3 &\rightarrow ID^3, \quad \overset{\Lambda}{a} \oplus \overset{\Lambda}{b} = \left(\vec{a} + \vec{b} \right) + \varepsilon \left(\vec{a}^* + \vec{b}^* \right) \\ \odot : ID \times ID^3 &\rightarrow ID^3, \quad \overset{\Lambda}{\lambda} \odot \overset{\Lambda}{a} = \lambda \vec{a} + \varepsilon \left(\lambda \vec{a}^* + \lambda^* \vec{a} \right) \end{aligned}$$

Algebraic construction $(ID^3, \oplus, ID, +, \cdot, \odot)$ is a modul. This modul is called *ID – Modul*.

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The inner product and vectorel product of dual vectors $\vec{\bar{a}}, \vec{\bar{b}} \in ID^3$ are defined by respectively,

$$\begin{aligned} \langle \cdot, \cdot \rangle &: ID^3 \times ID^3 \rightarrow ID, \quad \left\langle \vec{\bar{a}}, \vec{\bar{b}} \right\rangle = \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon \left(\left\langle \vec{a}, \vec{b}^* \right\rangle + \left\langle \vec{a}^*, \vec{b} \right\rangle \right) \\ \wedge &: ID^3 \times ID^3 \rightarrow ID^3, \quad \vec{\bar{a}} \wedge \vec{\bar{b}} = \left(\vec{a} \wedge \vec{b} \right) + \varepsilon \left(\vec{a} \wedge \vec{b}^* + \vec{a}^* \wedge \vec{b} \right) \end{aligned}$$

For $\vec{\bar{a}} \neq 0$, the norm $\left\| \vec{\bar{a}} \right\|$ of $\vec{\bar{a}} = \vec{a} + \varepsilon \vec{a}^*$ is defined by

$$\left\| \vec{\bar{a}} \right\| = \sqrt{\left\langle \vec{\bar{a}}, \vec{\bar{a}} \right\rangle} = \left\| \vec{a} \right\| + \varepsilon \frac{\left\langle \vec{a}, \vec{a}^* \right\rangle}{\left\| \vec{a} \right\|}, \quad \left\| \vec{a} \right\| \neq 0.$$

The angle between unit dual vectors $\vec{\bar{a}}$ and $\vec{\bar{b}}$ $\Phi = \varphi + \varepsilon \varphi^*$ is called dual angle and this angle is denoted by

$$\left\langle \vec{\bar{a}}, \vec{\bar{b}} \right\rangle = \cos(\Phi) = \cos(\varphi) - \varepsilon \varphi^* \sin(\varphi)$$

Let

$$\begin{aligned} \vec{\bar{\alpha}} &: I \subset IR \rightarrow ID^3 \\ s &\rightarrow \vec{\bar{\alpha}}(s) = \alpha(s) + \varepsilon \alpha^*(s) \end{aligned}$$

be differential unit speed dual curve in dual space ID^3 . Denote by $\{T, N, B\}$ the moving dual frenet frame along the dual space curve $\vec{\bar{\alpha}}(s)$ in the dual space ID^3 . Then T, N and B are the dual tangent, the dual principal normal and the dual binormal vector fields, respectively. The function $\kappa(s) = k_1 + \varepsilon k_1^*$ and $\tau(s) = k_2 + \varepsilon k_2^*$ are called dual curvature and dual torsion of $\vec{\bar{\alpha}}$, respectively. Then for the dual curve $\vec{\bar{\alpha}}$ the frenet formulae are given by,

$$(2.1) \quad \begin{cases} T'(s) = \kappa(s) N(s) \\ N'(s) = -\kappa(s) T(s) + \tau(s) B(s) \\ B'(s) = -\tau(s) N(s) \end{cases}$$

The formulae (2.1) are called the frenet formulae of dual curve in [9]. In this palace curvature and torsion are calculated by,

$$(2.2) \quad \kappa(s) = \sqrt{\langle T', T' \rangle}, \quad \tau(s) = \frac{\det(T, T', T'')}{\langle T', T' \rangle}$$

If α is not unit speed curve, then curvature and torsion are calculated by,

$$(2.3) \quad \kappa(s) = \frac{\|\alpha'(s) \wedge \alpha''(s)\|}{\|\alpha'(s)\|^3}, \quad \tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\|\alpha'(s) \wedge \alpha''(s)\|^2}$$

If formulae (2.1) is separated into the real and dual part, we can obtain

$$(2.4) \quad \begin{cases} t'(s) = k_1 n \\ n'(s) = -k_1 t + k_2 b \\ b'(s) = -k_2 n \end{cases}$$

$$(2.5) \quad \begin{cases} t^{*'}(s) = k_1 n^* + k_1^* n \\ n^{*'}(s) = -k_1 t^* - k_1^* t + k_2 b^* + k_2^* b \\ b^{*'}(s) = -k_2 n^* - k_2^* n \end{cases}$$

3. SOME CHARACTERIZATIONS INVOLUTE OF DUAL CURVES

Definition 1. Let $\alpha : I \rightarrow ID^3$ and $\beta : I \rightarrow ID^3$ be dual unit speed curves. If the tangent lines of the dual curve α is orthogonal to the tangent lines of the dual curve β , the dual curve β is called involute of the dual curve α or the dual curve α is called evolute of the dual curve β . According to this definition, if the tangent of the dual curve α is denoted by T and the tangent of the dual curve β is denoted by \bar{T} , we can write

$$(3.1) \quad \langle T, \bar{T} \rangle = 0$$

Theorem 1. Let α and β be dual curves. If the dual curve β involute of the dual curve α , we can write

$$\beta(s) = \alpha(s) + [(c_1 - s) + \varepsilon c_2] T(s), \quad c_1, c_2 \in \mathbb{R}.$$

Proof. Then by the definition we can assume that

$$(3.2) \quad \beta(s) = \alpha(s) + \lambda T(s), \quad \lambda(s) = \mu(s) + \varepsilon \mu^*(s)$$

For some function $\lambda(s)$. By taking derivate of the equation (3.2) with respect to s and applying the frenet formulae (2.1) we have

$$\frac{d\beta}{ds} = \left(1 + \frac{d\lambda}{ds}\right) T + \lambda \kappa N$$

where s and s^* are arc parameter of the dual curves α and β , respectively,

$$(3.3) \quad \bar{T} \frac{ds^*}{ds} = \left(1 + \frac{d\lambda}{ds}\right) T + \lambda \kappa N$$

Taking the inner product of (3.3) with T we have

$$(3.4) \quad \frac{ds^*}{ds} \langle T, \bar{T} \rangle = \left(1 + \frac{d\lambda}{ds}\right) \langle T, T \rangle + \lambda \langle T, N \rangle$$

By the defination we have

$$\left\langle T, \bar{T} \right\rangle = 0$$

By substituting the last equation in (3.4) we get

$$(3.5) \quad 1 + \frac{d\lambda}{ds} = 0 \text{ and } \frac{d}{ds}(\mu(s) + \varepsilon\mu^*(s)) = -1$$

The necessary operations are maken, we get

$$\mu'(s) = -1 \text{ and } \mu^{*'}(s) = 0$$

By taking the integral of the last equation we get

$$(3.6) \quad \mu(s) = c_1 - s \text{ and } \mu^*(s) = c_2$$

By substituting (3.6) in (3.2) we get

$$(3.7) \quad \beta(s) - \alpha(s) = [(c_1 - s) + \varepsilon c_2] T(s).$$

□

Corollary 1. *The distance between the dual curves β and α is $|c_1 - s| \mp \varepsilon c_2$.*

Proof. By taking the norm of the equation (3.7) we get

$$(3.8) \quad d(\alpha(s), \beta(s)) = |c_1 - s| \mp \varepsilon c_2$$

□

Theorem 2. *Let α, β be dual curves. If the dual curve β involute of the dual curve α , The relationships between the dual frenet vectors of the dual curves α and β*

$$\left\{ \begin{array}{l} \bar{T} = N \\ \bar{N} = -\cos\Phi T + \sin\Phi B \\ \bar{B} = \sin\Phi T + \cos\Phi B \end{array} \right.$$

Proof. By differentiating the equation (3.2) with respect to s we obtain

$$(3.9) \quad \beta'(s) = \lambda\kappa(s) N(s) , \quad \lambda = (c_1 - s) + \varepsilon c_2$$

and

$$\|\beta'(s)\| = \lambda\kappa(s)$$

Thus, the tangent vector of β is found

$$\bar{T} = \frac{\beta'(s)}{\|\beta'(s)\|} = \frac{\lambda\kappa(s) N(s)}{\lambda\kappa(s)}$$

If we arrange the last equation we obtain

$$(3.10) \quad \bar{T} = N(s)$$

By differentiating the equation (3.9) with respect to s we obtain

$$\beta'' = -\lambda\kappa^2 T + (\lambda\kappa' - \kappa) N + \lambda\kappa\tau B$$

If the cross product $\beta' \wedge \beta''$ is calculated we have

$$(3.11) \quad \beta' \wedge \beta'' = \lambda^2 \kappa^2 \tau T + \lambda^2 \kappa^3 B$$

The norm of vector $\beta' \wedge \beta''$ is found

$$(3.12) \quad \|\beta' \wedge \beta''\| = \lambda^2 \kappa^2 \sqrt{\kappa^2 + \tau^2}$$

For the dual binormal vector of the dual curve β we can write

$$\bar{B} = \frac{\beta' \wedge \beta''}{\|\beta' \wedge \beta''\|}$$

By substituting (3.11) and (3.12) in the last equation we get

$$(3.13) \quad \bar{B} = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B$$

For the dual principal normal vector of the dual curve β we can write

$$\bar{N} = \bar{B} \wedge \bar{T}$$

and

$$(3.14) \quad \bar{N} = -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B$$

□

Let Φ ($\Phi = \varphi + \varepsilon\varphi^*$, $\varepsilon^2 = 0$) be dual angle between the dual darbox vector W of α and dual unit binormal vector B in this situation we can write

$$(3.15) \quad \sin\Phi = \frac{\tau}{\kappa^2 + \tau^2}, \quad \cos\Phi = \frac{\kappa}{\kappa^2 + \tau^2}$$

By substituting (3.15) in (3.12) and (3.13) the proof is completed.

If the equation

$$\begin{cases} \bar{T} = N \\ \bar{N} = -\cos\Phi T + \sin\Phi B \\ \bar{B} = \sin\Phi T + \cos\Phi B \end{cases}$$

is separated into the real and dual part, we can obtain

$$\begin{cases} \bar{t} = n \\ \bar{n} = -\cos\varphi t + \sin\varphi b \\ \bar{b} = \sin\varphi t + \cos\varphi b \end{cases}$$

$$\begin{cases} \bar{t}^* = n^* \\ \bar{n}^* = -\cos\varphi t^* + \sin\varphi b^* + \varphi^* (\sin\varphi t + \cos\varphi b) \\ \bar{b}^* = \sin\varphi t^* + \cos\varphi b^* + \varphi^* (\cos\varphi t - \sin\varphi b) \end{cases}$$

On the way

$$\begin{cases} \sin\Phi = \sin(\varphi + \varepsilon\varphi^*) = \sin\varphi + \varepsilon\varphi^*\cos\varphi \\ \cos\Phi = \cos(\varphi + \varepsilon\varphi^*) = \cos\varphi - \varepsilon\varphi^*\sin\varphi \end{cases}$$

If the equation

$$\sin\Phi = \frac{\tau}{\kappa^2 + \tau^2}$$

is separated into the real and dual part, we can obtain

$$\begin{cases} \sin\varphi = \frac{k_2}{k_1^2 + k_2^2} \\ \cos\varphi = \frac{k_1^2 + k_2^* - 2k_1 k_2 k_1^* - 2k_2^2 k_2^*}{\varphi(k_1^2 + k_2^2)^2} \end{cases}$$

If the equation

$$\cos\Phi = \frac{\kappa}{\kappa^2 + \tau^2}$$

is separated into the real and dual part, we can obtain

$$\begin{cases} \cos\varphi = \frac{k_1}{k_1^2 + k_2^2} \\ \sin\varphi = \frac{2k_1^2 + k_1^* + 2k_1 k_2 k_2^* - k_1^2 k_1^* - k_2^2 k_1^*}{\varphi(k_1^2 + k_2^2)^2} \end{cases}$$

Theorem 3. *Let α, β be dual curves. If the dual curve β involute of the dual curve α , curvature and torsion of the dual curve β are*

$$(3.16) \quad \bar{\kappa}^2(s) = \frac{\kappa^2(s) + \tau^2(s)}{\lambda^2(s) \kappa^2(s)}, \quad \bar{\tau}(s) = \frac{\kappa(s) \tau'(s) - \kappa'(s) \tau(s)}{\lambda(s) \kappa(s) (\kappa^2(s) + \tau^2(s))}$$

Proof. By the defination of involute we can write

$$(3.17) \quad \beta(s) = \alpha(s) + |\lambda| T(s)$$

By differentiating the equation (3.17) with respect to s we obtain

$$\begin{aligned} \frac{d\beta}{ds^*} \frac{ds^*}{ds} &= T(s) + |\lambda|' T(s) + |\lambda| \kappa(s) N(s) \\ \frac{d\beta}{ds^*} \frac{ds^*}{ds} &= T(s) - T(s) + |\lambda| \kappa(s) N(s) \end{aligned}$$

$$(3.18) \quad \bar{T}(s) \frac{ds^*}{ds} = |\lambda(s)| \kappa(s) N(s)$$

Since the direction of $\bar{T}(s)$ is coincident with $N(s)$ we have

$$(3.19) \quad \bar{T}(s) = N(s)$$

Taking the inner product of (3.18) with T and necessary operation are maken we get

$$(3.20) \quad \frac{ds^*}{ds} = |\lambda(s)| \kappa(s)$$

By taking derivative of (3.19) and applying the frenet formulae (2.1) we have

$$(3.21) \quad \bar{T}(s) = N(s) \Rightarrow \bar{T}'(s) \frac{ds^*}{ds} = -\kappa T + \tau B$$

From (3.20) and (3.21) we have

$$\bar{T}'(s) = \frac{-\kappa T + \tau B}{|\lambda(s)| \kappa(s)}$$

From the last equation we can write

$$\bar{\kappa}(s) \bar{N}(s) = \frac{-\kappa T + \tau B}{|\lambda(s)| \kappa(s)}$$

Taking the inner product the last equation with each other we have

$$\left\langle \bar{\kappa}(s) \bar{N}(s), \bar{\kappa}(s) \bar{N}(s) \right\rangle = \left\langle \frac{-\kappa T + \tau B}{|\lambda(s)| \kappa(s)}, \frac{-\kappa T + \tau B}{|\lambda(s)| \kappa(s)} \right\rangle$$

Thus, we find

$$\bar{\kappa}^2(s) = \frac{\kappa^2(s) + \tau^2(s)}{\lambda^2(s) \kappa^2(s)}$$

We know that

$$\beta' \wedge \beta'' = \lambda^2 \kappa^2 \tau T + \lambda^2 \kappa^3 B$$

Taking the norm the last equation we get

$$\|\beta' \wedge \beta''\| = \kappa^4 \lambda^4 (\kappa^2 + \tau^2)$$

By substituting these equations in (2.3) we get

$$\bar{\tau} = \frac{\begin{vmatrix} 0 & \kappa\lambda & 0 \\ -\kappa^2\lambda & (\kappa\lambda)' & \kappa\tau\lambda \\ (-\kappa^2\lambda)' - \kappa(\kappa\lambda)' & -\kappa^3\lambda + (\kappa\lambda)'' - \kappa\tau^2\lambda & (\kappa\lambda)' \tau + (\kappa\tau\lambda)' \end{vmatrix}}{\|\beta' \wedge \beta''\|^2}$$

$$\bar{\tau} = \frac{\kappa\tau' - \kappa'\tau}{\kappa|\lambda|(\kappa^2 + \tau^2)}$$

□

If the equation (3.16) is separated into the real and dual part, we can obtain

$$\begin{cases} \bar{k}_1 = \frac{\sqrt{k_1^2 + k_2^2}}{\mu k_1} \\ \bar{k}_1^* = \frac{(\mu^2 k_1^2)(2k_1 k_1^* + 2k_2 k_2^*) - (2k_1 k_1^* \mu^2)(k_1^2 + k_2^2)}{2\mu^3 k_1^3 \sqrt{k_1^2 + k_2^2}} \end{cases}$$

$$\begin{cases} \bar{k}_2 = \frac{k_1 k_2' - k_2 k_1'}{\mu k_1 (k_1^2 + k_2^2)} \\ \bar{k}_2^* = \frac{(k_1 k_2'^* + k_2 k_1'^* - k_1' k_2^* - k_2 k_1'^*)(\mu k_1^3 + k_1 k_2^2 \mu) - [2(k_1 k_1^* + k_2 k_2^*) k_1 \mu + (k_1^2 + k_2^2)(k_1^* \mu + k_1 \mu^*)](k_1 k_2' - k_2 k_1')}{(\mu k_1^3 + k_1 k_2^2 \mu)^2} \end{cases}$$

Theorem 4. Let α, β be dual curves and the dual curve β involute of the dual curve α . If W and \bar{W} are darbox vectors of the dual curves α and β we can write

$$(3.22) \quad \bar{W} = \frac{1}{\lambda\kappa} (W + \Phi' N)$$

Proof. Since \bar{W} is darbox vector of $\beta(s)$ we can write

$$(3.23) \quad \bar{W}(s) = \bar{\tau}(s) \bar{T}(s) + \bar{\kappa}(s) \bar{B}(s)$$

By substituting $\bar{\tau}, \bar{T}, \bar{\kappa}, \bar{B}$ in the last equation we get

$$(3.24) \quad \bar{W}(s) = \frac{\kappa\tau' - \kappa'\tau}{\kappa|\lambda|(\kappa^2 + \tau^2)} N(s) + \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa|\lambda|} (\sin\Phi T + \cos\Phi B)$$

By substituting (3.15) in (3.24) we get

$$\bar{W}(s) = \frac{\kappa\tau' - \kappa'\tau}{\kappa|\lambda|(\kappa^2 + \tau^2)} N(s) + \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa|\lambda|} \left(\frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}} \right)$$

The necessary operation are maken, we get

$$\bar{W}(s) = \frac{\tau T + \kappa B}{\kappa|\lambda|} + \frac{\kappa\tau' - \kappa'\tau}{\kappa|\lambda|(\kappa^2 + \tau^2)} N(s)$$

$$\bar{W}(s) = \frac{1}{\kappa|\lambda|} \left(\tau T + \kappa B + \frac{\kappa\tau' - \kappa'\tau}{\kappa^2 + \tau^2} N \right)$$

and

$$\bar{W}(s) = \frac{1}{\kappa|\lambda|} \left(W + \frac{(\frac{\tau}{\kappa})' \kappa^2}{\kappa^2 + \tau^2} N \right)$$

Furthermore, Since

$$\frac{\sin\Phi}{\cos\Phi} = \frac{\tau/\sqrt{\kappa^2 + \tau^2}}{\kappa/\sqrt{\kappa^2 + \tau^2}}$$

$$\frac{\tau}{\kappa} = \tan\Phi$$

By taking derivative of the last equation we have

$$\Phi' \sec^2\Phi = \left(\frac{\tau}{\kappa}\right)'$$

The necessary operations are maken, we get

$$\Phi' = \left(\frac{\tau}{\kappa}\right)' \frac{\kappa}{\kappa^2 + \tau^2}$$

In this situation, the proof is completed

$$\bar{W}(s) = \frac{1}{\kappa|\lambda|} (W + \Phi' N)$$

□

If the equation (3.22) is separated into the real and dual part, we can obtain

$$\begin{cases} \bar{w} = \frac{w + \varphi' n}{\mu k_1} \\ \bar{w}^* = \frac{\mu k_1 (w^* + \varphi' n + \varphi'^* n) - (\mu k_1^* + \mu^* k_1) (w + \varphi' n)}{\mu^2 k_1^2} \end{cases}$$

If the equation (3.24) is separated into the real and dual part, we can obtain

$$\begin{cases} \bar{w} = \frac{\sqrt{k_1^2 + k_2^2}}{\mu k_1} (\sin\varphi t + \cos\varphi b) \\ \bar{w}^* = \frac{\sqrt{k_1^2 + k_2^2}}{\mu k_1} (\sin\varphi t^* + \cos\varphi b^* + \varphi^* (\cos\varphi t - \sin\varphi b)) + \frac{\mu k_1 (k_1 k_1^* + k_2 k_2^*) - (k_1^2 + k_2^2) (\mu k_1^* + \mu^* k_1)}{\sqrt{k_1^2 + k_2^2} \mu^2 k_1^2} (\sin\varphi t + \cos\varphi b) \end{cases}$$

Theorem 5. Let α, β be dual curves and the dual curve β involute of the dual curve α . If C and \bar{C} are unit vectors of the direction of W and \bar{W} , respectively

$$(3.25) \quad \bar{C} = \frac{\Phi'}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} N + \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} C$$

Proof. Since β the dual angle between \bar{W} and \bar{B} we can write

$$\bar{C}(s) = \sin\beta \bar{T}(s) + \cos\beta \bar{B}(s)$$

In here, we want to find the statements $\sin\beta$ and $\cos\beta$,

We know that

$$\sin\beta = \frac{\bar{\tau}}{\|\bar{W}\|} = \frac{\bar{\tau}}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}$$

By substituting $\bar{\tau}$ and $\bar{\kappa}$ in the last equation and necessary operations are taken, we get

$$(3.26) \quad \sin\beta = \frac{\Phi'}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}}$$

Similary,

$$(3.27) \quad \cos\beta = \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}}$$

Thus we find

$$\bar{C} = \frac{\Phi'}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} \bar{T} + \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{\Phi'^2 + \kappa^2 + \tau^2}} C$$

□

If the equation (3.25) is separated into the real and dual part, we can obtain

$$\left\{ \begin{array}{l} \bar{c} = \frac{\varphi' n + \sqrt{k_1^2 + k_2^2} c}{\sqrt{\varphi'^2 + k_1^2 + k_2^2}} \\ \bar{c}^* = \frac{\varphi' n^* + \varphi'^* n + \sqrt{k_1^2 + k_2^2} c^* + \frac{k_1 k_1^* + k_2 k_2^*}{\sqrt{k_1^2 + k_2^2}} c - \frac{\varphi' n (\sqrt{k_1^2 + k_2^2}) c (\varphi' \varphi'^* + k_1 k_1^* + k_2 k_2^*)}{\sqrt{\varphi'^2 + k_1^2 + k_2^2}}}{\sqrt{\varphi'^2 + k_1^2 + k_2^2}} \end{array} \right.$$

If the equation (3.26) and (3.27) are separated into the real and dual part, we can obtain

$$\left\{ \begin{array}{l} \sin\bar{\varphi} = \frac{\varphi'}{\sqrt{\varphi'^2 + k_1^2 + k_2^2}} \\ \cos\bar{\varphi} = \frac{(\Phi'^2 + \kappa^2 + \tau^2) \Phi'^* - \varphi' \varphi'^* + k_1 k_1^* + k_2 k_2^* \varphi'}{\bar{\varphi}^* (\Phi'^2 + \kappa^2 + \tau^2)^{\frac{3}{2}}} \end{array} \right. \quad \left\{ \begin{array}{l} \cos\bar{\varphi} = \sqrt{\frac{k_1^2 + k_2^2}{\varphi'^2 + k_1^2 + k_2^2}} \\ \sin\bar{\varphi} = \frac{(\varphi' \varphi'^* + k_1 k_1^* + k_2 k_2^*) \sqrt{k_1^2 + k_2^2} - (\varphi'^2 + k_1^2 + k_2^2) (k_1 k_1^* + k_2 k_2^*)}{\bar{\varphi}^* (\Phi'^2 + \kappa^2 + \tau^2)^{\frac{3}{2}} \sqrt{k_1^2 + k_2^2}} \end{array} \right.$$

Corollary 2. *Let α, β be dual curves and the dual curve β involute of the dual curve α . If evolute curve α is helix,*

- i) The vectors \bar{W} and \bar{B} of the involute curve β are linearly dependent.
- ii) $C = \bar{C}$
- iii) β is planar.

Proof. i) If the evolute curve α is helix, then we have

$$\frac{\tau}{\kappa} = \tan\Phi = \text{cons or } \Phi' = 0$$

and then we have

$$(3.28) \quad \begin{cases} \sin \bar{\Phi} = 0 \\ \cos \bar{\Phi} = 1 \end{cases}$$

Thus, we get

$$(3.29) \quad \bar{\Phi} = 0$$

ii) Substituting by the equation (3.29) into the equation (3.25) , we have

$$C = \bar{C}$$

iii) For being is a helix , then we have

$$\frac{\tau}{\kappa} = cons$$

$$(3.30) \quad \left(\frac{\tau}{\kappa} \right)' = 0$$

On the other hand, from the equation (3.16) , we can write

$$\frac{\bar{\tau}}{\bar{\kappa}} = \frac{\frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2 + \tau^2)}}{\frac{(\kappa^2 + \tau^2)^{\frac{1}{2}}}{\lambda\kappa}}$$

and

$$(3.31) \quad \frac{\bar{\tau}}{\bar{\kappa}} = \frac{\left(\frac{\tau}{\kappa} \right)' \kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}}$$

Substituting by the equation (3.30) into the equation (3.31) ,then we find

$$\bar{\tau} = 0$$

□

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